# Introduction to statistics

## Learning the basics of probability and statistics

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# Summary:

- Probability concepts
- Discrete distributions.
- Continuous distributions.
- Calculations on the Normal distribution.
- Convergence
- Inference

## Motivation

Figure 1: Dados apontam . . . (data shows . . . )

# Basic concepts of probability:

**Sample space $\Omega$**

It's the set of all the possible outcomes of a experiment, denoted by S or $\Omega$

**Event**

It's a subset of the sample space.

# Basic concepts of probability:

## Probability (Definition):

Given a experiment with a sample space $\Omega$ and a class of events $\mathcal{A}$, the probability denoted by $\mathbb{P}$ is a function which has $\mathcal{A}$ as domain and associate a numerical value between $[0,1]$ as image.

## Probability properties:

1. $\mathbb{P}(\Omega) = 1$ and $\mathbb{P}(\emptyset) = 0$
2. $0 \leq \mathbb{P}(A) \leq 1$, for every event A
3. For any sequence of mutually exclusive events $A_1, A_2, \ldots$ that's events that $A_i \bigcap A_j$ when $i \neq j$ we have that:

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(A_i)$$

# Basic concepts of probability:

## Event independence:

Two events are independent when the occurrence of the first does not affect the probability of ocurrence of the second.
Two events A and B are independent if:

$$\mathbb{P}(A \bigcap B) = \mathbb{P}(A)\mathbb{P}(B)$$

## Conditional Events:

The probability of a event A to occur given that the event B occurred is:

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \bigcap B)}{\mathbb{P}(B)}$$

# Basic concepts of probability:

**Bayes theorem:**

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(B|A)\mathbb{P}(A)}{\mathbb{P}(B)}$$

General case:

$$\mathbb{P}(A_i|B) = \frac{\mathbb{P}(B|A_i)\mathbb{P}(A_i)}{\sum_{j=1}^{n} \mathbb{P}(B|A_j)\mathbb{P}(A_j)}$$

**Bayes example (from Veritasium):**

You are felling sick, so you go to the doctor, there you run a battery of tests. After getting the results you tested positive for a rare disease (affects 0.1% of the population), the test will correctly identify that you have it 99% of the times.
What's the chances that you actually have the disease? 99%?

## Bayes example Solution

Let's denote the event of you have the disease H (stands for hypothesis, the prior) and the test been positive denoted by E (stands for evidence), so we have: $\mathbb{P}(H) = 0.001$ and $\mathbb{P}(E|H) = 0.99$

$$\mathbb{P}(H|E) = \frac{\mathbb{P}(E|H)\mathbb{P}(H)}{\mathbb{P}(E)} = \frac{\mathbb{P}(E|H)\mathbb{P}(H)}{\mathbb{P}(H)\mathbb{P}(E|H) + \mathbb{P}(H^C)\mathbb{P}(E|H^C)} =$$

$$= \frac{0.99 \cdot 0.001}{0.001 \cdot 0.99 + 0.999 \cdot 0.01} = 0.09 = 9\%$$

What if you test again and it's also positive? You can just take the posterior probability we just calculated and use as a prior:

$$= \frac{0.99 \cdot 0.09}{0.09 \cdot 0.99 + 0.91 \cdot 0.01} = 0.907 \approx 91\%$$

- Awesome video: A visual guide to Bayesian thinking

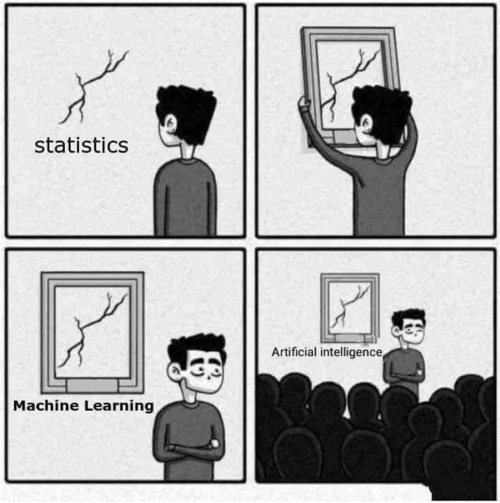



Figure 2: Credits: sandserifcomics

### Random Variable (RV)

Consider a experiment with a sample space $\Omega$ associated with it. A function that maps each element $\omega \in \Omega$ to a Real number such that $[w \leq X]$ it's called random variable (RV) $(X : \Omega \rightarrow \mathbb{R})$

- Example: Imagine a experiment that consist of 3 consecutive fair coin tosses, so the sample space of this experiment is: S = {(H,H,H), (H,H,T) , . . . (T,T,T)} . Now we want to create a random variable X that counts the number of heads in each outcome, so X((H,H,H)) = 3 and X((H,H,T)) = 2.

# Random Variable:

**Probability Mass Function (PMF):**

$$f_X(x) = \mathbb{P}[X = x] = \mathbb{P}[\{\omega \in \Omega : X(\omega) = x\}]$$

**Probability Density Function (PDF)**

$$\mathbb{P}[a \leq X \leq b] = \int_a^b f(x)dx$$

**Cumulative Distribution Function (CDF)**

$$F_X(x) = \mathbb{P}[X \leq x]$$

## Expectation:

- Discrete : $\mathbb{E}[X] = \sum x\mathbb{P}(X = x)$
- Continuous: $\mathbb{E}[X] = \int_{-\infty}^{\infty} xf(x)dx$

## Variance:

$$\mathbb{V}[X] = \sigma_X^2 = \mathbb{E}[X^2] - \mathbb{E}^2[X]$$

## Sample mean:

$$\overline{X_n} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

## Sample variance and standard deviation:

$$s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2$$

Standard deviation $= s$

# Discrete distributions

## Bernoulli:

Consider a experiment with has two possible outcomes: success (X=1, with probability p) or failure (X=0), this random variable is called Bernoulli, the PMF is:

$$\mathbb{P}(X = k) = p^k(1-p)^{1-k}$$

## Binomial:

Now consider a Bernoulli experiment conducted n times, let X be the random variable that represents the number of successes, X is called Binomial, the PMF is:

$$\mathbb{P}(X = k) = \binom{n}{k} p^k(1-p)^{n-k}$$

# Discrete distributions

## Geometric:

Again consider a Bernoulli experiment conducted n times, but the first n-1 are failures and the last nth is a success. Let X be number of tries , which is called Geometric, the PMF is:

$$\mathbb{P}(X = k) = (1 - p)^k p$$

- A important property is that Geometric distribution is the only discrete distribution that is **memoryless**.

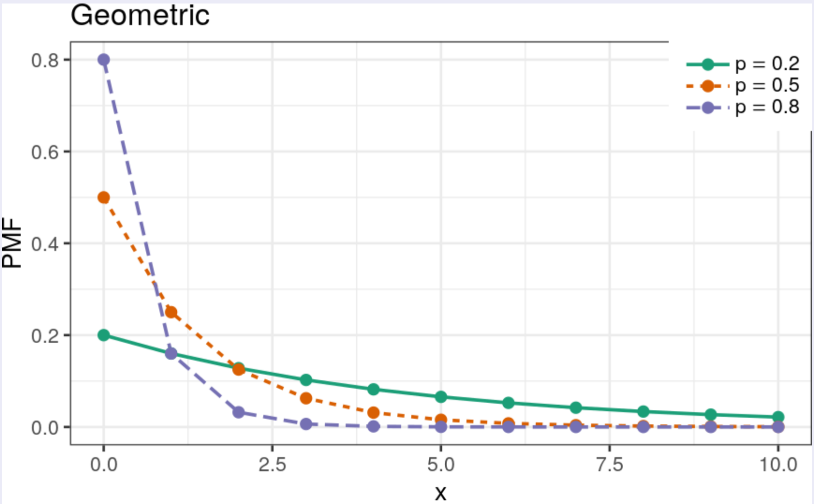
## Poisson:

A random variable which value can assume 0,1,2 ... is called Poisson with $\lambda > 0$ parameter if your PMF is:

$$\mathbb{P}(X = k) = \frac{e^{-\lambda}\lambda^k}{k!}$$

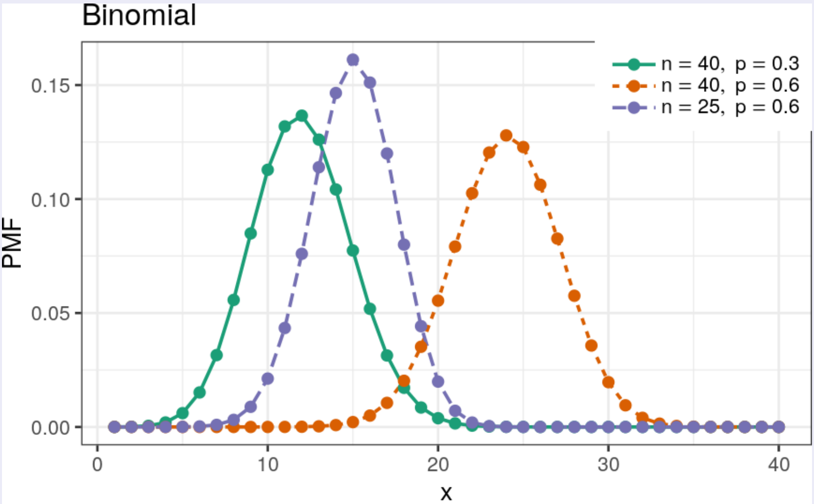
# Discrete distributions plots

## Geometric:



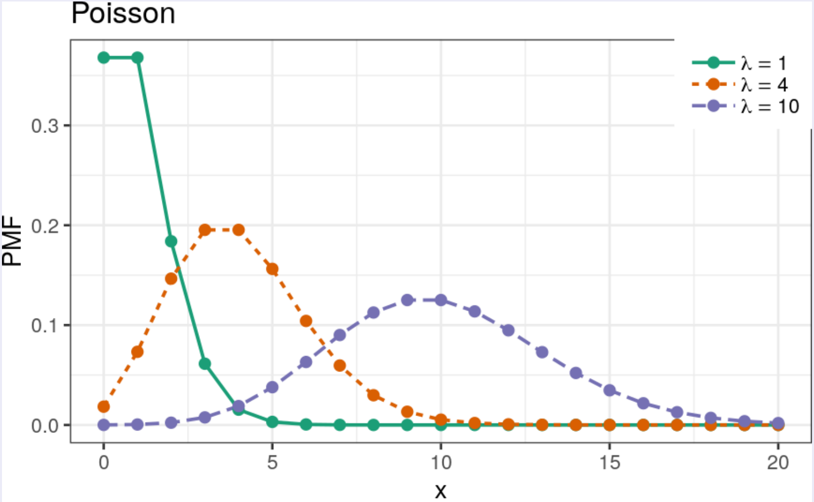
# Discrete distributions plots

## Binomial:



# Discrete distributions plots

Poisson:



# Continuous distributions

## Normal (or Gaussian, bell curve):

A continuous real random variable is called Normal with $\sigma^2 > 0$ (squared scale), $\mu \in \mathbb{R}$ (location) parameters if your PDF is:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right)$$

- The normal function is a example of Liouville's theorem, an probability cannot be analytically calculated, only be numeric methods.
- Fun facts: the half inside the exponential is for the variance to be 1, and the $\sqrt{2\pi}$ is for the integral in the whole support to become 1.

# Continuous distributions

## Exponential

A continuous **positive** random variable is called Exponential with $\lambda > 0$ (rate or inverse scale) parameter if your PDF is:

$$f(x) = \lambda e^{-\lambda x}$$

Important property: Exponential and Geometric (discrete) distribution are the only distributions that are **memoryless**.

## Memoryless property:

$$\mathbb{P}[X > x + y \mid X > y] = \mathbb{P}[X > x]$$

So no matter how much time has passed it's like the process is starting from beginning.

# Continuous distributions

## Pareto

A continuous $x \in [x_m, \infty)$ random variable is called pareto with $x_m > 0$ (scale) and $\alpha > 0$ (shape) parameters if your PDF is:

$$f(x) = \frac{\alpha x_m^\alpha}{x^{\alpha+1}}$$

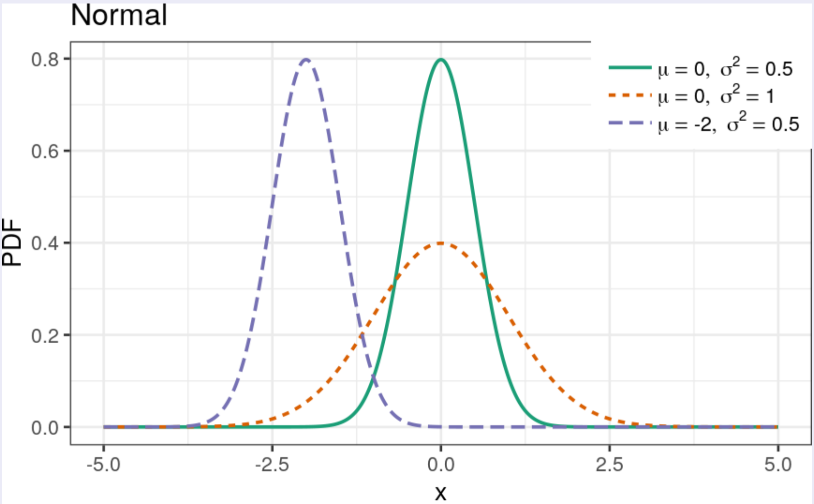
Zipf is the discrete distribution of pareto
Pareto is a **heavy tailed** distribution: It means it goes to zero slower (than exponential).

## Pareto principle (80-20 law):

The pareto principle states that 80% of results is caused by 20% of the effects, for example wealth distribution, software bugs etc ...
It's a particular pareto distributed values when $\alpha \approx 1.161$

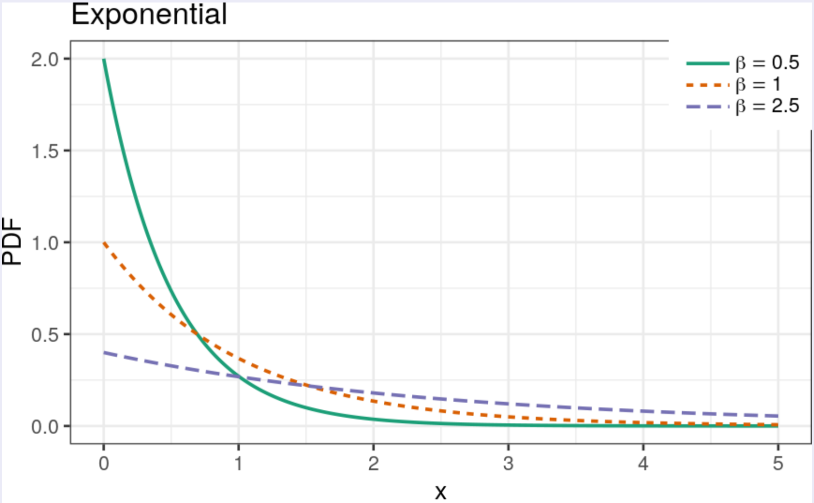
# Continuous distributions plots

## Normal:



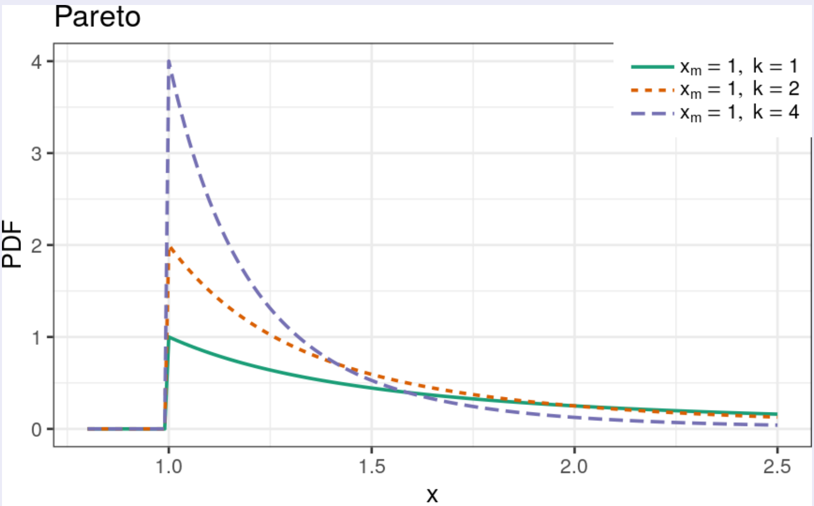
# Continuous distributions plots

## Exponential:



# Continuous distributions plots

Pareto:



## Calculations on the Normal distribution

Given a Normal distributed values, how to calculate the probability on it?

With normal distribution we usually use a standard normal (where $\mu = 0, \sigma = 1$) cumulative table and standardize the values.

- How to standardize the values: Given $X \sim N(\mu, \sigma^2)$

$$z = \frac{x - \mu}{\sigma} \quad \text{or} \quad z = \frac{x - \overline{x}}{s}$$

z is called **z score** and is **standard normal** distributed.

- Standard cumulative $\Phi(x)$:

$\Phi(x) = \mathbb{P}(z \leq x)$ also $\Phi(-x) = 1 - \Phi(x)$

$\Phi(x)$ values can we found in a table or using NORMSDIST function in Excel or in Python using stats.norm.cdf function from SciPy.

## 68-95-99.7 rule:

The 68-95-99.7 rule also know as the empirical rule is a shorthand to remember the percentage of Normal distributed values that lie within arround the mean with a width of 1,2,3 standard deviations.

$$\mathbb{P}(\mu - 1\sigma \leq X \leq \mu + 1\sigma) \approx 0.6827$$
$$\mathbb{P}(\mu - 2\sigma \leq X \leq \mu + 2\sigma) \approx 0.9545$$
$$\mathbb{P}(\mu - 3\sigma \leq X \leq \mu + 3\sigma) \approx 0.9973$$

We don't have to memorize this values, we can calculate it:

$$\mathbb{P}(\mu - 1\sigma \leq X \leq \mu + 1\sigma) = \mathbb{P}(-1\sigma \leq X - \mu \leq 1\sigma) =$$

$$\mathbb{P}\left(-1 \leq \frac{X-\mu}{\sigma} \leq 1\right) = \mathbb{P}(-1 \leq z \leq 1) = \Phi(1) - \Phi(-1) \approx 0.6827$$

## 68-95-99.7 rule: Chart

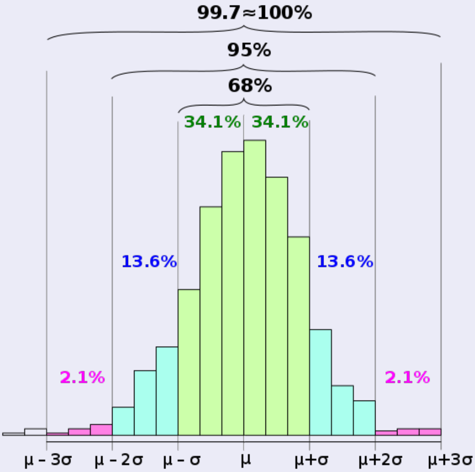
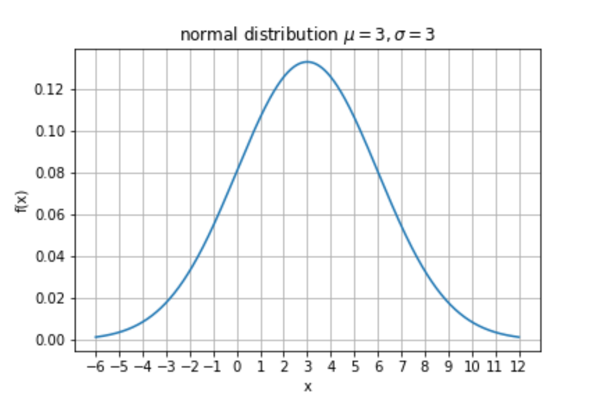


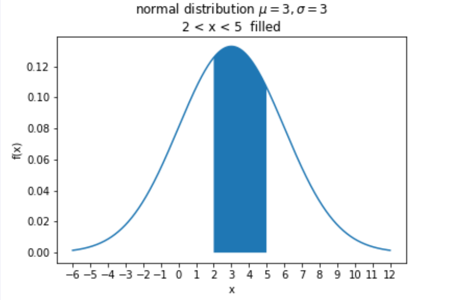

Figure 3: Credits: Wikipedia: Empirical rule histogram

## Calculations on the Normal distribution: Example from Ross:

X is a normal random variable with parameters: $\mu = 3$ and $\sigma^2 = 9$,
Calculate: (a) $\mathbb{P}(2 < X < 5)$ (b) $\mathbb{P}(X > 0)$ (c) $\mathbb{P}(|X - 3| > 6)$



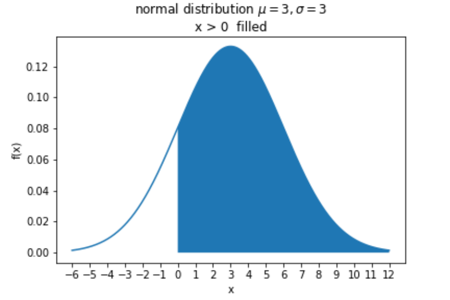
## Calculations on the Normal distribution: Example from Ross: part a




$$\mathbb{P}(2 < X < 5) = \mathbb{P}\left(\frac{2-\mu}{\sigma} < \frac{X-\mu}{\sigma} < \frac{5-\mu}{\sigma}\right) = \mathbb{P}\left(\frac{-1}{3} < Z < \frac{2}{3}\right) =$$

$$\Phi\left(\frac{2}{3}\right) - \Phi\left(\frac{-1}{3}\right) = \Phi\left(\frac{2}{3}\right) - \left[1 - \Phi\left(\frac{1}{3}\right)\right] \approx 0.3779$$

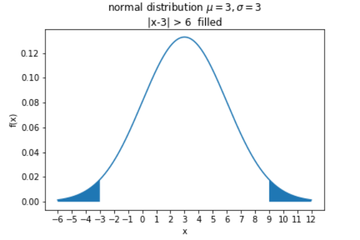
## Calculations on the Normal distribution: Example from Ross: part b




$$\mathbb{P}(X > 0) = \mathbb{P}\left(\frac{X - \mu}{\sigma} > \frac{-\mu}{\sigma}\right) = \mathbb{P}(Z > -1) = 1 - \Phi(-1) =$$

$$= \Phi(1) \approx 0.8413$$

# Calculations on the Normal distribution: Example from Ross: part c




$$\mathbb{P}(|X-3|>6) = \mathbb{P}(6 < X-3 < -6) = \mathbb{P}(9 < X < -3) =$$

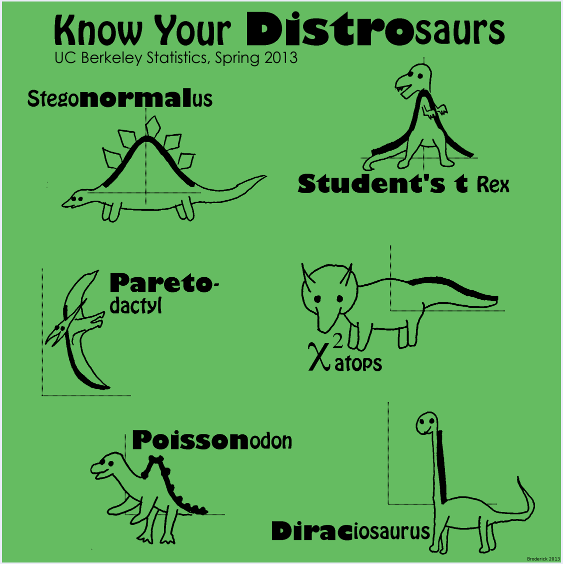
$$\mathbb{P}(X>9)+\mathbb{P}(X<-3) = \mathbb{P}\left(\frac{X-\mu}{\sigma} > \frac{9-\mu}{\sigma}\right)+\mathbb{P}\left(\frac{X-\mu}{\sigma} < \frac{-3-\mu}{\sigma}\right)$$

$$= \mathbb{P}(Z>2)+\mathbb{P}(Z<-2) = 2[1-\Phi(2)] \approx 0.0456$$

## Assumptions on distribution choice


Figure 4: Credits: Portal data science

## Assumptions on distribution choice



## Assumptions on distribution choice

In order to know which distribution of your data values understanding the nature of the problem is fundamental.
Is your values a result from counting? So is it discrete ? Or continuous? Which values are possible?

## Discrete distributions:

- Bernoulli: boolean result, example: coin toss, second turn (with only 2 candidates) election.
- Binomial: Number of “success” results given a permanent experiment runs. Example: From 20 devices after a long time what’s the probability of 15 of them has a kind of defect.
- Geometric: Number of failures until the first success. Example: The probability of winning the lottery is 1 in 1 million, What’s the probability of winning it after 3 tries?
- Poisson: Example: Number of cars on the road.

# Assumptions on distribution choice

**Continuous distributions:**

- Normal: No restriction on possible values (positive and negative values are valid). Example: The height of children of the same sex and age.
- chi-squared: Only positive values, unlike normal is not symmetric.
- Exponential: Only positive values, describes the time until failure.
- Pareto: Only positive values and bigger and $x_m$. Example: Size of gold mines, very few big mines and a lot of small ones.

## order statistics and quantiles

- Given $X_1, X_2, \cdots, X_n$ values from the same distribution, let:

$X_{(1)}$ the smallest value from $X_1, X_2, \cdots, X_n$ (minimum)
$X_{(2)}$ 2th smallest value from $X_1, X_2, \cdots, X_n$
$X_{(j)}$ jth smallest value from $X_1, X_2, \cdots, X_n$
$X_{(n)}$ the **biggest** value from $X_1, X_2, \cdots, X_n$ (maximum)

- q-quantiles are values that partition the values into q subsets of (almost) equal sizes. For instance: q=2 we have the median, 4 the quartiles , 100 percentile and so on ...
- Why use quantiles? Why use median instead of the mean? Because Order statistics is a **robust statistic** which means it is not affected by outliers.

## boxplots:

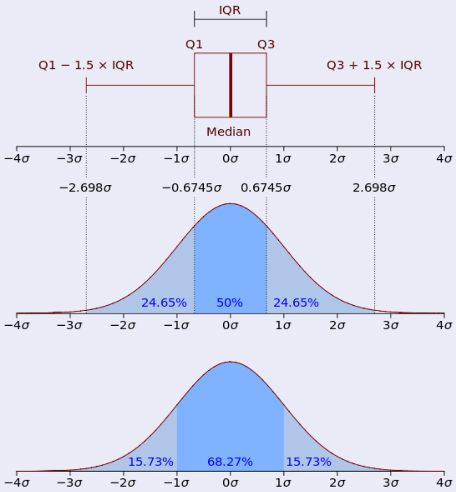
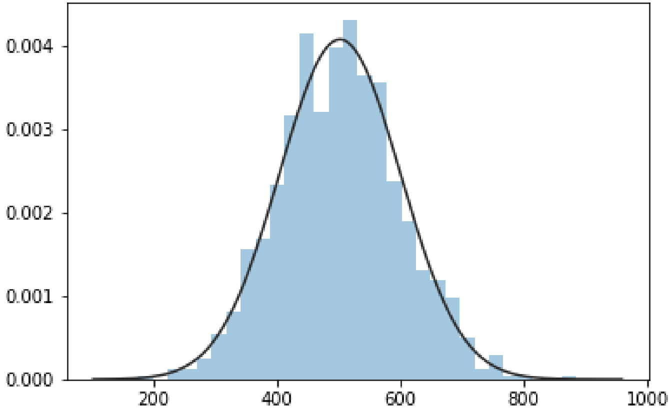


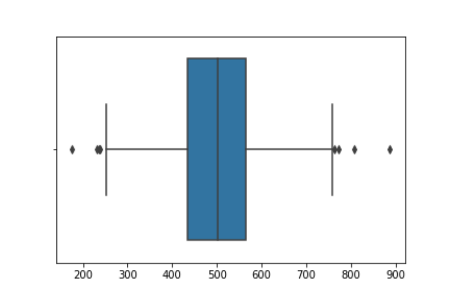

Figure 5: Credits: Wikipedia: Boxplot vs PDF

## Boxplots example:

Let X_n be a sequence of normal distributed random variable with $\mu = 500$ and $\sigma = 100$ we have n=1000 samples, results:



## Boxplots example:




| | |
|---|---|
| count | 1000.000000 |
| mean | 501.933206 |
| std | 97.921594 |
| min | 175.873266 |
| 25% | 435.240969 |
| 50% | 502.530061 |
| 75% | 564.794388 |
| max | 885.273149 |

## Convergence

In statistics there some types of convergence, the main ones are: Let $\{X_1, X_2, \cdots\}$ be a sequence of identically distributed random variables.

1. In Probability: $X_n \xrightarrow{p} Y$ :

$$(\forall \varepsilon > 0) \quad \lim_{n\to\infty} \mathbb{P}(|X_n - Y| > \varepsilon) = 0$$

2. In distribution (weakly, in law): $X_n \xrightarrow{D} Y$

$$\lim_{n\to\infty} F_{X_n}(x) = F_Y(y)$$

3. Almost sure (strongly) : $X_n \xrightarrow{as} Y$

$$\mathbb{P}\left(\lim_{n\to\infty} X_n = Y\right) = 1$$

# Convergence

**Law of large numbers (LLN):**

Let $\{X_1, X_2, \cdots\}$ be a sequence of identically distributed random variables and $\mathbb{E}[X] = \mu$

**Weak (WLLN)**

$\overline{X_n} \xrightarrow{p} \mu \quad n \to \infty$

**Strong (SLLN)**

$\overline{X_n} \xrightarrow{as} \mu \quad n \to \infty$

In words: The sample mean converge to the (theoretical) expected value as the sample size increases.

# Convergence

### Central Limit Theorem (CLT)

Let $\{X_1, X_2, \cdots\}$ be a sequence of identically distributed random variables and $\mathbb{E}[X] = \mu$ and $\mathbb{V}[X] = \sigma^2$

The CLT states that:

$$\overline{X_n} \xrightarrow{D} \mathcal{N}\left(\mu, \frac{\sigma^2}{n}\right)$$

After some transformations we have:

$$\frac{\sqrt{n}(\overline{X_n} - \mu)}{\sigma} \xrightarrow{D} \mathcal{N}(0, 1)$$

### Inference:

Inference is the process to deduce (estimate) the properties of underlying probability distribution from the data values.

### Maximum likelihood estimator (MLE):

One of the most popular estimator method is the maximum likelihood estimator, it consists of finding the parameters that maximizes the likelihood function (via derivatives).

### Q-Q plot:

Q-Q (quantile-quantile) plot: Is a plot where data quantiles are ploted in one axis and the theoretical quantiles of the fitted distribution on the other axis and a linear regression of the points. This plot can be used to provide a assessment of the “goodness of fit” and maybe find out the data outliers on the data.

### Inference process

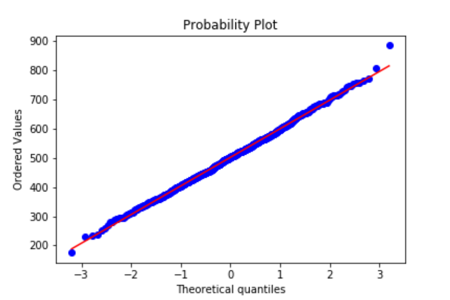
Given some data the inference process goes like this:

1. Choose a distribution (based on the nature of the problem).
2. Fit the distribution (estimate the parameters).
3. Create a q-q plot to judge the goodness of fit, if some outliers are found they can be identified (and maybe left out).

If the line still not good try starting again with a different distribution.

## Inference example:

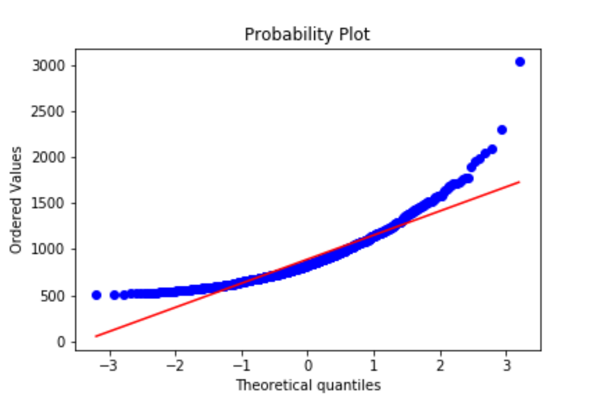
Using the same data from the boxplot example, the Q-Q plot:




The plot has created using stats.probplot from SciPy.
Using stats.norm.fit we can estimate (fit) the parameters (assuming normal distribution), we get: loc=501.93 and scale=97.87 we generated random values using loc=500,scale=100n=1000 (the fit could get better with a bigger sample)

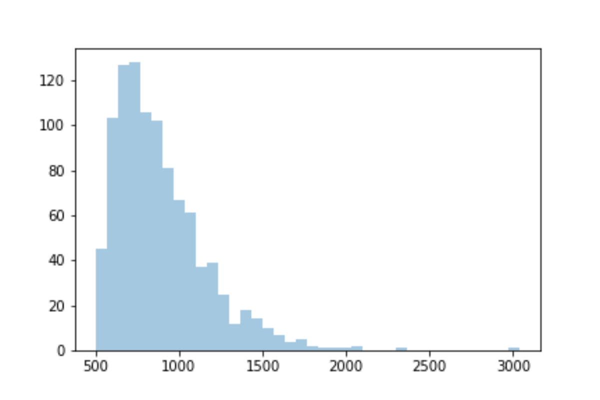
## Inference example 2:

With new data (n=1000), the Q-Q plot:



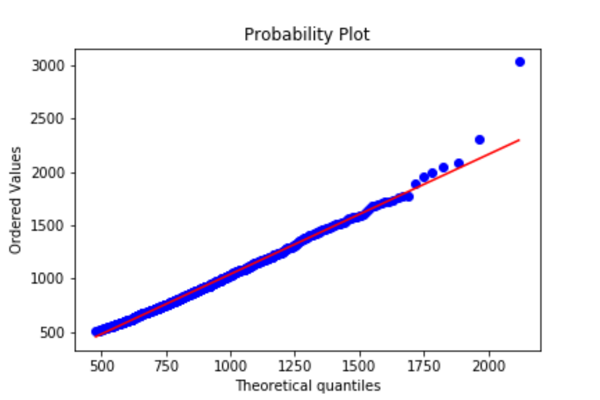

Looking at the line we can clearly see that the line did not fit the points, the data probably is not normal distributed.

## Inference example 2: histogram:




the values is not symmetric, one good guest is that it's a chi squared distribution.

## Inference example 2: Q-Q plot assuming chi squared:




Using stats.chi2.fit we can estimate (fit) the parameters: df=5.5,loc=460,scale=74, we generated random values using df=4,loc=500,scale=100.

## Further reading:

To study those topics in depth, here are some awesome references:

### Getting started:

- Podcast: (pt) Pizza de Dados
- Book: The Lady Tasting Tea: (pt) Uma Senhora Toma Chá.
- /r/dataisbeautiful
- Scipy lectures scientific examples with Python.

### Studying material:

- (pt) Portal Action
- The Probability and Statistics Cookbook
- Havard Statistics 110: Probability
- Statistics and probability
- Random website
- Book: (en) First Course in Probability by Sheldon Ross: (pt) Probabilidade: Um Curso Moderno com Aplicações